

# An Explicit Method to Write Belyĭ Morphisms

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## Abstract

In this work, for an infinite family of embedded graphs on the sphere we explicitly calculate the corresponding Belyĭ morphisms. The importance of the family relies on the fact that the associated triangulation is non-negatively curved, and hence is canonically an element of the lattice  $W$ . Thurston has found in [6]. This settles a non-trivial relationship in between. We, further, propose a method to deal with curves of higher genus, emphasizing hyperelliptic curves.

## 1 Introduction

It is a well-known fact that the absolute Galois group,  $\text{Gal}(\mathbf{Q})$  acts on graphs embedded into surfaces, and further such objects define a complex structure on the surface making it into an algebraic curve. Such algebraic curves correspond to pairs  $(X, \beta)$ ; where  $X$  is an algebraic curve, and  $\beta$  is a meromorphic function on  $X$  which ramifies at most over 3 points, may be chosen without loss of generality as  $\{0, 1, \infty\}$ . It seems to be a quite hard task to understand the action of  $\text{Gal}(\mathbf{Q})$  on such pairs. Hence we restrict ourselves to certain sub-objects. Here, We will deal with algebraic curves which have certain automorphisms. Namely, we will begin with the elliptic curve  $y^2 = x^3 - 1$ , then we will use curves which have automorphisms of order  $p$ ; where  $g = (p - 1)/2$ , to write Belyĭ morphisms explicitly.

## 2 Preliminaries

In this section, to fix notation, we will state some well-known facts.

### 2.1 Hypergeometric Differential Equation

We start with a homogeneous linear ordinary differential equation of order 2:

$$\frac{d^2\omega}{dx^2} + p(x)\frac{d\omega}{dx} + q(x)\omega = 0 \quad (1)$$

where  $p(x)$  and  $q(x)$  are functions of the complex variable  $x$ , i.e. the equation (1) will be considered as defined over the Riemann sphere,  $\mathbf{P}^1$ . A point  $x = x_0$  is called a *singular point* of the equation (1) if  $p$  or  $q$ , or both, have a pole at  $x_0$ .  $x_0$  is called a *regular singular point* of (1), when the function  $p$  has at most a pole of order 1, and the function  $q$  has at most a pole of order 2 at  $x_0$ ; and the equation is called *Fuchsian*, exactly when all the singular point are regular. The first *non-trivial* Fuchsian differential equation occurs

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when it possesses 3 regular singular points, i.e. when we have a *hypergeometric differential equation*. It is not so hard to see that any hypergeometric differential equation can be put into the following form, in suitable coordinates:

$$\frac{d^2\omega}{dx^2} + \left[ \frac{1-\lambda}{x} + \frac{1-\mu}{x-1} \right] \frac{d\omega}{dx} + \left[ \frac{(1-\lambda-\mu)^2 - \nu^2}{4x(x-1)} \right] \omega = 0 \quad (2)$$

where, the numbers  $\lambda, \mu, \nu$  are referred to as the *exponent differences*. The solutions of (2) exist, and are linearly independent, by the fundamental theorem of Cauchy ([7, Section 2.2]), and are given by hypergeometric series. Throughout we will name these solutions  $\omega_1$  and  $\omega_2$ .

It was observed by Schwarz that the behaviour of the quotient  $y(x) = \frac{\omega_1(x)}{\omega_2(x)}$  is very special. More precisely, one can show

**Proposition 2.1.** ([2]) *Any branch of the function  $y(x)$  maps  $\mathbf{C}$  to two neighbouring triangles in*

- i.  $\mathbb{H}$  (considered as equipped with its usual hyperbolic structure, when  $\lambda + \mu + \nu < 1$ )
- ii.  $\mathbf{C}$  (considered with its euclidean structure, when  $\lambda + \mu + \nu = 1$ )
- iii.  $\mathbf{P}^1$  (considered with its spherical metric, when  $\lambda + \mu + \nu > 1$ )

with angles  $\lambda\pi, \mu\pi$  and  $\nu\pi$ .

## 2.2 Triangle Groups and Dessin d'Enfants

A *dessin d'enfant* is a bipartite graph,  $\Gamma$ , i.e a connected CW complex comprised of 0 and 1 dimensional cells, embedded in an oriented, not necessarily closed, surface  $S$  with:

- i. the embedding,  $\iota$ , from  $\Gamma$  to  $S$  is injective
- ii. each connected component of the set  $S \setminus \iota(\Gamma)$ , which are called *faces*, is homeomorphic to an open disc

Note that since we work on an oriented surface, the embedding  $\iota$  gives us an orientation around every vertex, i.e. a numbering of the edges around every vertex. Hence if one chooses a marking of the graph, that is a numbering of the edges, then one can write a subgroup, called the *cartographic group*, of the symmetric group on  $|\{\text{edges of } \Gamma\}|$  letters whose generators are rotations around black vertices, call  $\sigma$ , rotations around white vertices, call  $\tau$ .

This is the point where triangle groups come into the picture. Recall that a triangle group of signature  $(k, l, m)$ ,  $k, l, m \in \mathbf{Z}_{>0}$  has the following presentation

$$(k, l, m) := \langle \sigma, \tau \mid \sigma^k = \tau^l = (\sigma \cdot \tau)^m = 1 \rangle$$

Putting the above two ideas together, we arrive at the conclusion that one may write a group epimorphism from a suitably chosen triangle group to the cartographic group of a dessin. On the other hand, when all the numbers,  $1/\lambda, 1/\mu, 1/\nu$ , see Equation 2, are non-negative integers, the inverse,  $x(y)$ , of the function  $y(x) = \frac{\omega_1}{\omega_2}(x)$  is naturally automorphic with respect to the triangle group of signature  $(1/\lambda, 1/\mu, 1/\nu)$ , which is a consequence of Proposition 2.1.

Our main idea will be to use a very specific elliptic curve and elliptic functions together with the hypergeometric differential equation to find the corresponding Belyı morphism, where a Belyı morphism is defined as a rational function on the curve associated with  $S$  which is ramified over at most 3 points.

### 3 The Family $D_n$ and the Corresponding Belyĭ Functions

In this section, we are going to use the facts that we stated earlier to calculate the Belyĭ function of a particular family  $D_n$ .

#### 3.1 The Family, $D_n$ , of Dessins

It is known that to every dessin one can associate a triangulation and vice versa when the triangles in the triangulation can be colored in two colors, say black and white, so that no two neighbouring triangles, i.e. triangles that have an edge in common, have the same color. Our family will be described via a family of triangulations of the sphere.

One can describe the family  $D_n$  in terms of a family of triangulations  $T_n$  as follows: take a (equilateral) triangle, name it  $T_1$ . Inside  $T_1$ , draw another triangle whose edges connect the midpoints of the original triangle (this new triangle is upside down), as in (Figure 1) to obtain the second element,  $T_2$ .

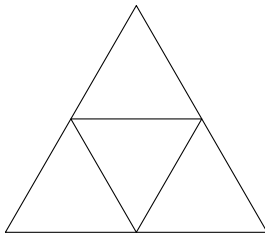


Figure 1: Second element,  $T_2$ , of the family.

The next case is a simple generalization of the previous one. Instead of dividing the edges into two, we will divide into three, and insert the appropriate edges to obtain a triangulation as in (Figure 2).

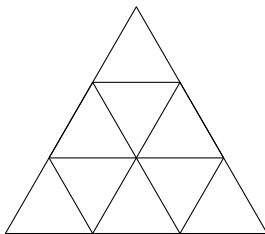


Figure 2: Third element,  $T_3$ , of the family.

To obtain the  $n^{\text{th}}$  element of the family, we take  $n - 1$  equally distributed points on its each edge and then insert the appropriate edges to get a triangulation of the triangle. To obtain a triangulation,  $S_n$ , of the sphere, take two copies of  $T_n$ , and glue them *only* from the boundary to obtain a sphere. For example,  $S_1$  is simply just two triangles glued from their boundary, hence the triangulation we get is the following:

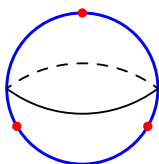
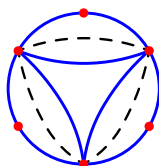
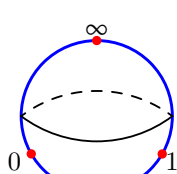


Figure 3: The very first element,  $S_1$ , of the family

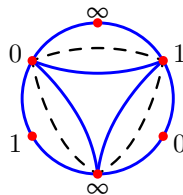
To obtain the next element of the sphere triangulation,  $S_2$ , we glue two copies of  $T_2$  and get:



Now, let's label the vertices of the triangulation to determine the corresponding dessin.



(a) Labelling  $S_1$



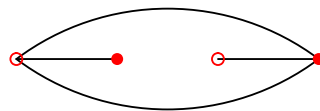
(b) Labelling  $S_2$

Figure 4: Labeling the vertices of triangulations

We, deliberately, forget about vertices labeled with  $\infty$  and edges incident to such vertices. Denote with "white" vertices the inverse image of 0, and finally with "black" vertices the inverse image of 1 in order to obtain the associated dessins:



(a) The dessin corresponding to  $S_1$



(b) The dessin corresponding to  $S_2$

Figure 5: First two dessins

### 3.2 Calculating the Belyĭ Morphism

Now, we have a family of dessins on the sphere. Name the dessin corresponding to  $S_n$  as  $D_n$ . We would like to know the Belyĭ function associated to each  $D_n$ . The first is immediate: the identity function, when we normalize the white vertex to be at  $z = 0$  and white vertex to be at  $z = 1$ . We normalize the second as: the rightmost black vertex is at  $z = \alpha$ , the next white vertex is at  $z = 0$ . We set  $z = \beta$  for the leftmost white vertex, and the remaining vertex is set to be at  $z = 1$ . For convenience, we mark the pole in the middle, as  $z = \gamma$  (see the following Figure):

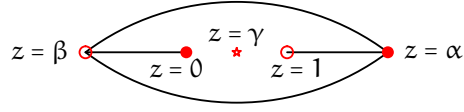


Figure 6: Normalization of  $S_2$ .

Then, the Belyĭ map must satisfy the equality:

$$k \frac{(z-1)(z-\beta)^3}{(z-\gamma)^3} = k \frac{z(z-\alpha)^3}{(z-\gamma)^3} + 1 \quad (3)$$

whose solution determines the four unknowns, namely  $k = \frac{1}{8}$ ,  $\alpha = 2$ , and  $\beta = -1$ , and finally  $\gamma = \frac{1}{2}$ . It is known from the general theory, [5], that the equations obtained from the equality of the two polynomials, as we did, determines the Belyĭ morphism. However, even for  $S_3$ , it is "almost" impossible to solve the system of equations. At this point, in order to be able to pour in the tools that we introduced in the beginning, we make some observations.

We start with:

**Lemma 3.1.** *The cartographic group  $C_n$  of  $D_n$  admits an epimorphism from the triangle group with signature  $(3,3,3)$ , whose kernel is torsion-free.*

*Proof.* By the definition of our family, the generator of the cartographic group have order 3 because each cycle has 3 elements. Hence sending each generator of  $C_n = \langle \sigma, \tau \rangle$ , to a suitable element in the triangle group with signature  $(3,3,3)$ , denoted by  $\Delta$ . Since the orders of the elements match, there cannot be any element of finite order in kernel.  $\square$

In what follows, the elliptic curve  $y^2 = x^3 - 1$  will be denoted by  $E$ . It is well-known that the curve,  $E$  is the only elliptic curve with an automorphism which has a fixed point of order 6. Moreover, one can write a multi-valued map, say  $\pi$ , from the quotient  $\Delta \backslash \mathbf{C} \cong \mathbf{P}^1$  to  $E$ . It is easy to describe the covering map,  $\pi$ , if one uses a hexagon instead of a parallelogram to denote the fundamental domain of  $E$ :

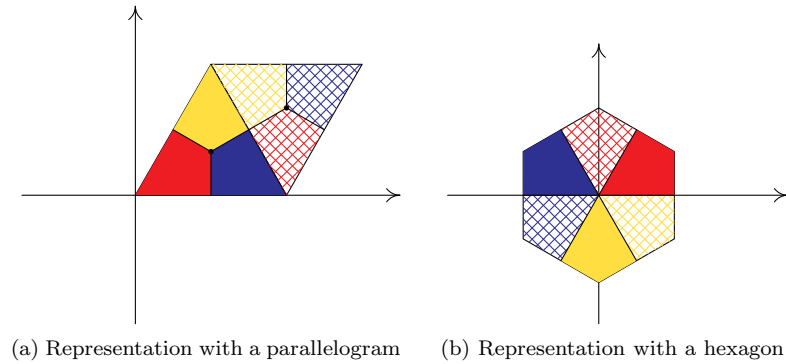


Figure 7: Fundamental regions corresponding to  $E$

The hexagon picture enables us to write the canonical projection we mentioned:

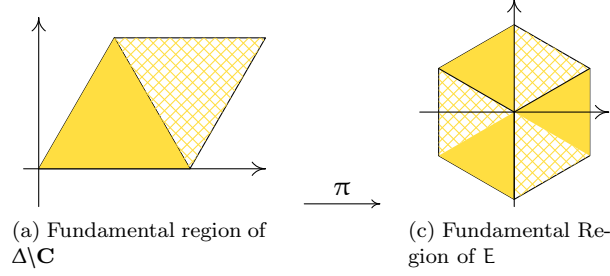


Figure 8: Geometric Description of the map  $\pi$

So after solving Equation 2 for the triangle group  $\Delta$ , i.e. for the exponential differences  $\lambda = \frac{1}{3}$ ,  $\mu = \frac{1}{3}$  and  $\nu = \frac{1}{3}$ , we obtain two functions  $\omega_1$  and  $\omega_2$ . Then, by Proposition 2.1 the map  $x(y)$ ; where  $y(x) = \frac{\omega_1}{\omega_2}(x)$  is automorphic with respect to  $\Delta$ . It is also well known that the function field of  $E$  is generated by the *Weierstraß functions*,  $\wp$  and  $\wp'$ . Using the fact that rational functions on  $\Delta \setminus \mathbf{C}$  are uniquely determined by their zeros and poles, in our case  $x(y) = \frac{\omega_1}{\omega_2}(y)$  on  $\Delta \setminus \mathbf{C}$ , one immediately sees that the function  $\varphi(z) = \frac{-4\sqrt{-1}}{\wp'(z)-2\sqrt{-1}}$  is the correct one which gives us the following commutative diagram:

$$\begin{array}{ccc} \Delta \setminus \mathbf{C} & \xrightarrow{(\omega_1/\omega_2)^{-1}} & \mathbf{P}^1 \\ \downarrow \pi & & \uparrow \varphi \\ E & \xrightarrow{m_1} & E \end{array}$$

where we the map  $m_1$  denotes the identity map. More generally, consider the degree  $i$  self-map,  $m_i$ , sending each point  $z \in E$  to  $i \cdot z \in E$ , for  $i \in \mathbf{N}$ . Observe that the function  $\varphi$ , as well as  $\wp'$ , is invariant under the action of the lattice  $\mathbf{Z}[\exp(2\pi\sqrt{-1}/6)]$ . Hence the composition  $\varphi \circ m_i$  is invariant with respect to  $\frac{1}{i}\mathbf{Z}[\exp(2\pi\sqrt{-1}/6)]$ . If one pulls back the triangulation on  $E$  induced on  $E$  from the quotient  $\Delta \setminus \mathbf{C}$  using the map  $m_i$  instead of  $m_1$ , since the map is “linear”, one obtains a tiling of the fundamental parallelogram of  $E$ . In particular, when  $i = 2$ , we have:

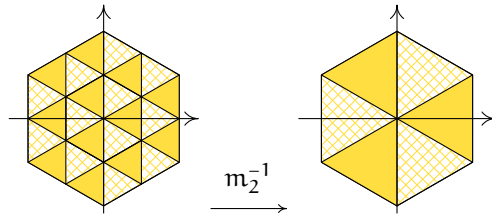


Figure 9: The map  $m_2$  geometrically

In fact, for each positive integer  $i$ , the triangulation obtained by pulling back via  $m_i$  the natural triangulation of  $E$ , and the triangulation obtained by choosing equally distributed  $i - 1$  points on the edges of the triangulation, and completing the configuration to a triangulation of  $E$  are *same*. We conclude:

**Proposition 3.2.** *The pull-back of the triangulation to  $\Delta \setminus \mathbf{C}$  via  $\pi$  is nothing but  $S_i$ .*

*Proof.* We recall that any pair of neighbouring triangles represents a fundamental region for  $\Delta$ . Hence we have the result.  $\square$

Just to give an idea, and to persuade the reader, we complete Figure 9:

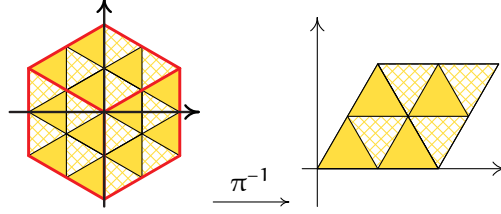


Figure 10: Image via  $\pi^{-1}$

So, what we are interested in is values of the function  $\varphi$  on the vertices of the triangulation.

*Remark.* Observe that these vertices are the, so called, *division points* of the curve. But to calculate the Belyi morphism, we do not need all *division values*, because the image of the fundamental region of  $\Delta$  is exactly one of the regions that is highlighted by red lines in Figure 10. Moreover, one can use the equivalence of points to simplify the calculations.

Before formulating the general case, we will demonstrate an example.

**Example 3.1.** We continue to the case  $i = 2$ . Let  $v_1 = 1/2$  and  $v_2 = \exp(2\pi\sqrt{-1}/6)/2$  in  $\mathbf{C}$  be the two *half-generators* for the lattice,  $\Lambda_E = \mathbf{Z}[\zeta_6]$ , corresponding to the elliptic curve  $E$ ; that is, they are half-periods of  $\Lambda_E$  (up to normalization, of course). Then what we only need is the values of  $\varphi$  at the points  $z_1 = v_1$ ,  $z_2 = \frac{v_1+v_2}{3}$  and  $z_3 = \frac{1}{3}(v_2 + (v_2 - v_1))$ , as indicated in the next figure:

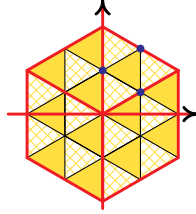


Figure 11: The points that interests us

These values can be simply calculated using the well-known formulas of elliptic functions, [3]. In our case, we obtain  $\varphi(v_1) = 2$ ,  $\varphi(\frac{v_1+v_2}{3}) = -1$  and  $\varphi(\frac{1}{3}(v_2 + (v_2 - v_1))) = \frac{1}{2}$ . Finally, let us choose at which edge Belyi morphism attains 0, 1 and  $\infty$ .

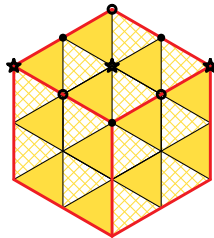


Figure 12: Distribution of  $0 = o$ ,  $1 = \bullet$  and  $\infty = *$

Summing all the things up we write our Belyĭ morphism up to constant factor  $k$  as:

$$\begin{aligned}
f_2(x) &= k \frac{(x - \varphi(\frac{v_1+v_2}{3}))^3 (x - \varphi(\frac{2(v_2+(v_2-v_1))}{3}))}{(x - \varphi(\frac{(v_2+(v_2-v_1))}{3}))^3} \\
&= k \frac{(x+1)^3 (x - \varphi(-\frac{2(v_1+v_2)}{3}))}{(x - \frac{1}{2})^3} \\
&= k \frac{(x+1)^3 (x - \frac{-4\sqrt{-1}}{\wp'(-\frac{2(v_1+v_2)}{3}) - 2\sqrt{-1}})}{(x - \frac{1}{2})^3} \\
&= k \frac{(x+1)^3 (x - \frac{-4\sqrt{-1}}{-2\sqrt{-1}-2\sqrt{-1}})}{(x - \frac{1}{2})^3} \\
&= k \frac{(x+1)^3 (x-1)}{(x - \frac{1}{2})^3} \text{ (compare with Equation 3)}
\end{aligned}$$

where we assume that the pole at  $v_1 + v_2$  is  $\infty$  so that our function is rational. To recover the function, the only remaining element is  $k$ , which is in this case  $\frac{1}{f_2(0)} = \frac{1}{8}$ .

*Remark.* To carry the calculations to the general case note also that by our construction, the dessin  $D_n$  is symmetric with respect to the *real axis*. So, it is enough to consider the division values in half of the red region, as shown in Figure 12 in the special case  $i = 2$ . This is also a property of  $\wp'$ : it attains the same values, hence the function  $\varphi$ . So, to describe the general case we will restrict our attention to only one “triangle”.

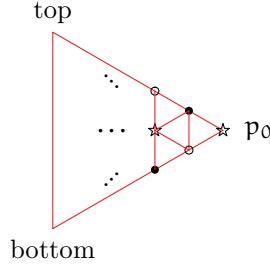


Figure 13: Distribution of ramification points.

Fix a positive integer  $i$ , and consider our main triangle corresponding to  $D_i$ . Set  $a \equiv i \pmod{3}$ , with  $a \in \{0, 1, 2\}$ . By looking at the values of  $a$  we may decide the type of the remaining two vertices of our triangle, which we will refer to as *top* and *bottom*. In particular, when  $a = 0$ , both the top and the bottom vertices will be in the inverse image of  $\infty$ ; when  $a = 1$ , then the top vertex will be in the inverse image of  $1$ , and bottom in the inverse image of  $0$ ; and finally when  $a = 2$ , then top will be in  $f_i^{-1}(0)$  and bottom in  $f_i^{-1}(1)$ . Let the sets  $Z_i$ ,  $O_i$ ,  $P_i$  denote the subset of our triangle whose elements are zeros, ones, poles of our embedded graph, respectively, except the three vertices  $p_0$ , top and bottom, see Figure 13. On the elliptic curve  $E$ , these points, i.e. the elements of the sets  $Z_i$ ,  $O_i$  and  $P_i$ , are  $i$ -division points. Set the rightmost pole,  $p_0$  to be  $\infty$ . Then the Belyĭ function,  $f_i$  of  $D_i$  can be written, up to the constant  $k$ , as:

$$f_i(x) = k \frac{\prod_{\zeta \in Z_i \setminus \mathbb{R}} ((x - \varphi(\zeta))^3 (x - \varphi(\bar{\zeta}))^3) \cdot \prod_{\zeta \in Z_i \cap \mathbb{R}} (x - \varphi(\zeta))^3}{\prod_{\zeta \in P_i \setminus \mathbb{R}} (x - \varphi(\zeta))^3 (x - \varphi(\bar{\zeta}))^3 \cdot \prod_{\zeta \in P_i \cap \mathbb{R}} (x - \varphi(\zeta))^3} \cdot g_i(x);$$



where  $R$  denotes the image of the real line under the projection  $\pi$ , which is the boundary of the triangle above and  $g_i(x)$  depends on the integer  $a$ . One can clarify the dependence to the integer  $a$  as follows:

$$g_i(x) := \begin{cases} \frac{1}{(x-\varphi(\text{top}))(x-\varphi(\text{bottom}))}, & \text{if } a = 0 \\ (x - \varphi(\text{bottom})), & \text{if } a = 1 \\ (x - \varphi(\text{top})), & \text{if } a = 2 \end{cases}$$

## 4 A Generalization

Note the following:

**Proposition 4.1.** *Let  $p \geq 5$  be a prime and  $X$  be a curve of genus  $g = (p-1)/2$ . Suppose that  $X$  has an automorphism of order  $p$ , say  $\sigma$ , or equivalently  $p \nmid \# \text{Aut}(X)$ . Then, the natural projection  $\pi_\sigma : X \rightarrow X/\langle \sigma \rangle$  is a Belyi morphism. In fact, it ramifies exactly over 3 points.*

*Proof.* Let  $g$  and  $g'$  denote the genus of  $X$  and the quotient  $X/\langle \sigma \rangle$ , respectively. By Riemann-Hurwitz formula we have:

$$\begin{aligned} 2g - 2 &= p(2g' - 2) + m(p-1) \\ (p-1) - 2 &= p(2g' - 2) + m(p-1) \end{aligned}$$

If we consider the above equation modulo  $p-1$  then

$$\begin{aligned} -2 &\equiv p(2g' - 2) \pmod{p-1} \\ 0 &\equiv g' \pmod{p-1} \end{aligned}$$

On the other hand, we must have  $g' < g = (p-1)/2$ , hence  $g' = 0$ . Plug in  $g' = 0$  in the above equation to get  $3(p-1) = m(p-1)$ , and hence the result.  $\square$

In terms of triangle groups, we may conclude the following:

**Corollary 1.** *Let  $X$  be an algebraic curve of genus  $g$  admitting an automorphism of order  $p = 2g+1$ , where  $p \geq 5$  is a prime number, hence  $g \geq 2$ . Then the surface group of  $X$  is a subgroup of the triangle group of signature  $(p, p, p)$ .*

For a finite group  $G$ , let  $M_g^G$  denote the moduli space of genus  $g$  curves,  $X$ , with the property that  $G \leq \text{Aut}(X)$ . Then one can rewrite Proposition 4.1 as follows: Let  $p \geq 5$  be an odd prime of the form  $(g-1)/2$  such that  $p|G$  and suppose that  $[X] \in M_g^G$ . Then there is an element  $\sigma \in \text{Aut}(X)$  whose order is  $p$ , and the natural projection  $X \rightarrow X/\langle \sigma \rangle$  is a Belyi morphism. In particular, any point in  $M_g^G$  corresponds to a curve  $X$  of genus  $g$  defined over  $\overline{\mathbf{Q}}$ .

Our first concern is the case when  $g = 2$ , and thus  $p = 5$ , i.e. when  $X$  is a hyperelliptic curve. An example is the curve with affine equation  $y^2 = x(x^5 - 1)$ . This curve has reduced automorphism group, which is the automorphism group of an hyperelliptic curve modulo the hyperelliptic involution,  $\mathbf{Z}/5\mathbf{Z}$ , [1, Section 11.7].

**A Closer Look at Hyperelliptic Curves.** Consider the following:

**Example 4.1.** Let  $E$  be the elliptic curve  $y^2 = x^3 - 1$ , and  $X$  be the hyperelliptic curve defined by the equation  $(y')^2 = (x')^6 - 1$ . Consider the map,  $\pi$ , sending  $x'^2$  to  $x$  and  $y'$  to  $y$ . Then  $\pi$  is of degree 2, ramified over  $x = 0$ , and  $x = \infty$ . Hence the composition  $\beta \circ \pi$  gives us the Belyi morphism corresponding to  $X$ ; where  $\beta$  is the Belyi morphism described geometrically in Figure 14.

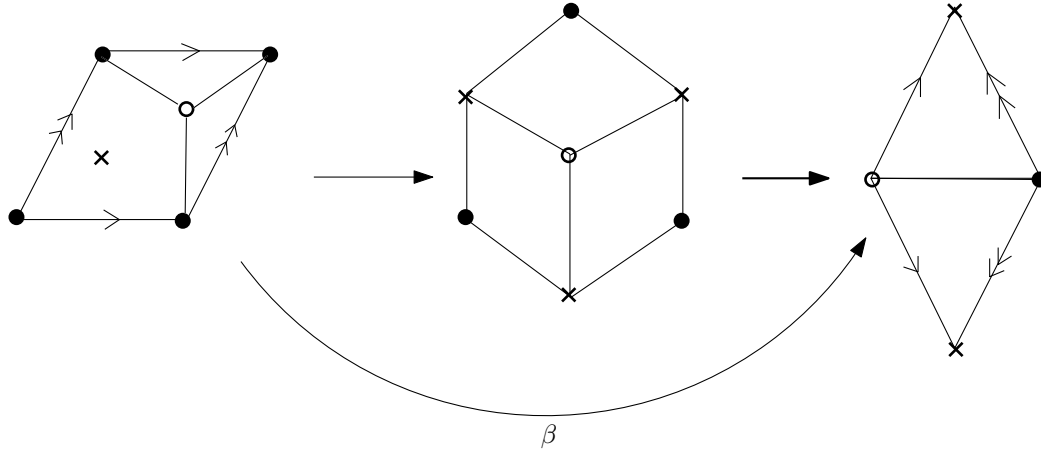


Figure 14: Dessin on  $y^2 = x^3 - 1$  induced by the inclusion  $\mathbf{Z}[\exp 2\pi i \sqrt{-1}/6] \leq \Delta_{3,3,3}$

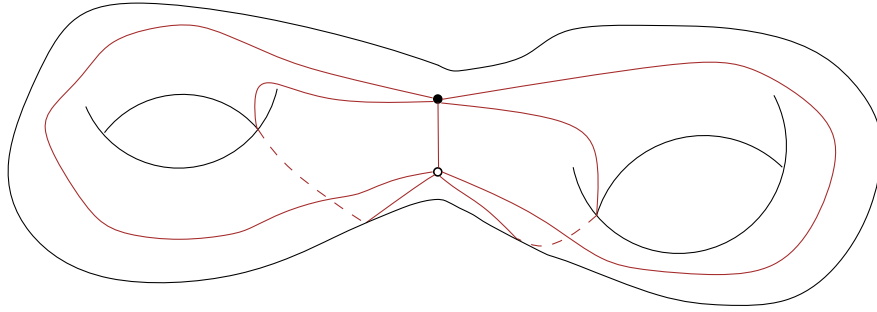


Figure 15: Dessin on  $y^2 = x(x^5 - 1)$

Example 4.1 is an application of a well-known method described as follows: if one can arrange the ramification points properly, then composition induce Belyĭ morphisms. Namely, let a curve  $E$  be given with a Belyĭ morphism  $\beta : E \rightarrow \mathbf{P}^1$ , and another curve  $X$  with a morphism  $\gamma : X \rightarrow E$  whose ramification *values* are exactly the points on  $E$  on which  $\beta$  ramifies. Then the composition  $\beta \circ \gamma : X \rightarrow \mathbf{P}^1$  is a Belyĭ morphism on  $X$ . However, the existence of such morphisms is an important question since one requires a precise ramification data. Now, in the described construction, let  $E$  be an elliptic curve defined over  $\overline{\mathbf{Q}}$ , hence it comes with a Belyĭ morphism  $\beta_E$ ,  $X$  be a hyperelliptic curve of genus two, which covers  $E$ . Then Riemann Hurwitz formula tells us, independent of the degree of the cover, that the ramification divisor has degree 2.

Case 1. If the divisor is  $2P$ , then by using the translation on  $E$ , we can send this point to the point  $\beta_E^{-1}(0)$ , to obtain the associated Belyĭ morphism. In this case, an information on the field of definition of  $X$  can also be obtained: say our morphism is of degree  $d$ ; where  $d$  is an odd prime, and let  $K$  be the number field over which the  $d$ -torsion points of  $E$  are defined. Then by [4, Corollary 1.3], the Hurwitz space functor of such covers are finely represented by a quasi-projective  $K$ -surface which implies that such covers may be defined over  $K$ .

Case 2. However, when the divisor is  $P_1 + P_2$ , one may not be able to arrange the morphism from  $X$  to  $E$  so that the image of ramification points under the composition fall into  $\mathbf{P}_{\overline{\mathbf{Q}}}^1$ .

**Example 4.2** (Klein's Quartic). We would like to demonstrate one more example concerning the case when  $g = 3$ , and hence  $p = 7$ . A well-known and at the same time well-studied example is Klein's quartic,  $X_k$ , whose equation is  $x^3y + y^3z + z^3x = 0$ . In this case, as a consequence of Corollary 1, the surface group  $S$  of  $X_k$  is included in the triangle group with signature  $(7, 7, 7)$ . The corresponding automorphism is the one which sends  $x$  to  $\zeta_7x$ ,  $y$  to  $\zeta_7^2y$  and  $z$  to  $\zeta_7^4z$ ; where  $\zeta_7$  is any primitive root of unity.

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